Universal solutions for the classical dynamical Yang–Baxter equation and the Maurer–Cartan equations



2004 J. Phys. A: Math. Gen. 37 453

(http://iopscience.iop.org/0305-4470/37/2/014)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 453–468 PII: S0305-4470(04)64781-0

# Universal solutions for the classical dynamical Yang–Baxter equation and the Maurer–Cartan equations*

**Emanuela Petracci**

Université de Cergy-Pontoise, Département de Mathématiques, UFR Sciences et Tecniques, 2 av. A Chauvin, 95302 Cergy-Pontoise Cedex, France

E-mail: petracci@math.u-cergy.fr

Received 15 June 2003, in final form 22 September 2003
Published 15 December 2003
Online at stacks.iop.org/JPhysA/37/453 (DOI: 10.1088/0305-4470/37/2/014)

**Abstract**

Using functional equations we solve the Maurer–Cartan equations and a special version of the classical dynamical Yang–Baxter equation (vCDYBE). Our solutions are valid for any Lie algebra over a base ring containing $\mathbb{Q}$, and in the case of vCDYBE for any quadratic Lie algebra. Our method applies also to Lie superalgebras.

PACS numbers: 02.20.Sv, 02.30.Ik, 02.40.Hw

## 1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional, real (or complex) Lie algebra. With any subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ and $\epsilon \in \mathbb{R}$ (or $\mathbb{C}$) is associated a classical dynamical Yang–Baxter equation (CDYBE) with coupling constant $\epsilon$ (see, for example, [6]). This equation is important in mathematics and physics (see, for example, [2]).

In [1] Alekseev and Meinrenken consider the case $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{g}$ a Lie algebra equipped with an invariant, non-degenerate, symmetric bilinear form (such a Lie algebra is called *quadratic*). They show that when $\mathfrak{g}$ is a compact Lie algebra, the analytic function

$$f(t) = -\frac{1}{t} + \frac{1}{2}\coth\left(\frac{t}{2}\right) \tag{1}$$

provides a solution for CDYBE with $\epsilon = \frac{1}{4}$. This solution is called the Alekseev–Meinrenken dynamical $r$-matrix.

* This research was done during a visit to the 'Institut de Mathématiques de Jussieu' in Paris, and completed during a visit—supported by the Swiss National Science Foundation—to the 'Section de Mathématiques' of the University of Geneva.

Here we show that $f(t)$ provides a solution when $\mathfrak{g}$ is any quadratic Lie superalgebra. In fact, when $\mathfrak{g} = \mathfrak{h}$ and $\mathfrak{g}$ is quadratic, CDYBE is equivalent to equation (21) which we call vCDYBE. It is an equation involving formal differential 3-forms on $\mathfrak{g}$, and it is defined for any Lie superalgebra over a superring $\mathbb{K} = \mathbb{K}_0 \oplus \mathbb{K}_1$ equipped with an even, invariant, bilinear form $\gamma$ (we do not suppose that $\gamma$ is non-degenerate or symmetric, nor that $\mathfrak{g}$ is finitely generated). We show that each odd formal power series $f \in \mathbb{K}_0[[t]]$ verifying

$$\begin{aligned}\frac{f(u+t)-f(u)}{t} &+ \frac{f(u+v)-f(v)}{u} + \frac{f(v+t)-f(t)}{v} + \frac{f(u+t)-f(t)}{u} \\ &+ \frac{f(v+u)-f(u)}{v} + \frac{f(t+v)-f(v)}{t} \\ &= 2(f(t)f(u)+f(u)f(v)+f(v)f(t)+\epsilon) \quad \text{modulo} \quad u+t+v \end{aligned} \tag{2}$$

gives a solution of vCDYBE. When $\mathbb{K} \supseteq \mathbb{Q}$ and $\epsilon = \frac{1}{4}$, (1) is the unique odd solution of (2).

In [7] Fehér and Pusztai give, for an ordinary Lie algebra, another direct proof of the fact that $f$ is a solution of vCDYBE. If we put $v = -t - u$ in (2), we get the functional equation used in [7]. Even in the case of Lie algebras, our proof is more elementary, it reduces to the definition of a Lie (super)algebra (and, in particular, to *Jacobi's identity* (9)).

We show that vCDYBE has strong similarities with Maurer–Cartan equations.

Let $\mathbb{K} = \mathbb{K}_0 \oplus \mathbb{K}_1$ be a commutative superring containing $\frac{1}{2}$, and $\mathfrak{g}$ a Lie superalgebra over $\mathbb{K}$. We consider a formal differential 1-form $\tilde{\alpha}$ on $\mathfrak{g}$ with values in $\mathfrak{g}$, and we denote by $\mathrm{d}\tilde{\alpha}$ its de Rham differential. One of the Maurer–Cartan equations is

$$\mathrm{d}\tilde{\alpha} = \tfrac{1}{2}[\tilde{\alpha}, \tilde{\alpha}]. \tag{3}$$

We show that each formal power series $f \in \mathbb{K}_0[[t]]$ verifying

$$\frac{f(u+t)-f(u)}{t} + \frac{f(u+t)-f(t)}{u} = f(u)f(t)$$

provides a solution of equation (3). When $f(0) = 1$ this solution is called a Maurer–Cartan form. In this case, a solution exists only when $\mathbb{K} \supseteq \mathbb{Q}$, and in this situation, there exists a unique solution given by $f(t) = \frac{\mathrm{e}^t - 1}{t}$.

Maurer–Cartan equations are important in the theory of Lie groups. For example, they may be used to prove the third Lie theorem (see remark 5.1). Variations of Maurer–Cartan equations now occur in many places, as in the deformation theory and in the Weiss–Zumino–Novikov–Witten model.

This paper is organized as follows. We consider arbitrary Lie superalgebras over an arbitrary commutative superring containing $\frac{1}{2}$. To introduce Maurer–Cartan equations (section 5) and vCDYBE (section 6.2) we need the notion of formal differential form on such a Lie superalgebra. In section 2 we define the notion of formal differential form on a $\mathbb{Z}/2\mathbb{Z}$-graded module and its de Rham complex, and in section 3 we define a Lie superalgebra. To solve our equations we use the formulae in section 4. To get these formulae we need some tools on Lie superalgebras that are described in section 3. In section 6.2 we also explain the relation between vCDYBE and CDYBE.

## 2. Modules over a superring

### *2.1. Generalities*

We recall the basic definitions and examples which we will use to define a differential form. They are from super linear algebra (see, for example, [8]).

We say that $\mathbb{K}$ is a *superring* if it is a unitary ring graded over $\mathbb{Z}/2\mathbb{Z}$. For each non-zero homogeneous element $a \in \mathbb{K}$ we denote by $p(a)$ its degree. We say that $a$ is even if $p(a) = 0$, and that $a$ is odd if $p(a) = 1$.

The superring $\mathbb{K}$ is called *commutative* if $ab = (-1)^{p(a)p(b)}ba$ for all homogeneous and non-zero $a, b \in \mathbb{K}$; and $a^2 = 0$ when $a$ is odd.

**Convention 2.1.** *Each time we use the symbol $p(a)$ for an element $a$ of a $\mathbb{Z}/2\mathbb{Z}$-graded group occurring in a linear expression, it is implicitly assumed that it is non-zero and homogeneous. Moreover, the expression is extended by linearity. For example, the expression above will be written as $ab = (-1)^{p(b)p(a)}ba$ for any $a, b \in \mathbb{K}$.*

*Each time we consider a graded group $M$, we denote by $M_0$ and $M_1$ the subgroups composed of elements with even and odd degree.*

*From now to the end of this text, $\mathbb{K}$ will be a commutative superring.*

**Definition 2.1.** *A commutative group $(M, +)$ graded over $\mathbb{Z}/2\mathbb{Z}$ is a $\mathbb{K}$-module if it is equipped with a bilinear application $M \times \mathbb{K} \to M$, such that $(m\alpha)\beta = m(\alpha\beta)$ and $p(m\alpha) = p(m) + p(\alpha)$, for any $\alpha, \beta \in \mathbb{K}$ and $m, n \in M$.*

*If $\mathbb{K}$ is a field, $M$ is also called a $\mathbb{K}$-supervector space.*

**Definition 2.2.** *Let $M, N$ be two $\mathbb{K}$-modules. A map $F : M \to N$ is a morphism of $\mathbb{K}$-modules if $F(m\alpha) = F(m)\alpha$ for any $m \in M$ and $\alpha \in \mathbb{K}$. We also say that $F$ is $\mathbb{K}$-linear.*

We denote by $\mathrm{Hom}(M, N)$ the group of functions $F : M \to N$ which are morphisms of $\mathbb{K}$-modules. It is graded over $\mathbb{Z}/2\mathbb{Z}$ in the following way: $F$ is even if $F(M_0) \subseteq N_0$ and $F(M_1) \subseteq N_1$, and $F$ is odd if $F(M_0) \subseteq N_1$ and $F(M_1) \subseteq N_0$.

$\mathrm{Hom}(M, N)$ is a $\mathbb{K}$-module: for any $F \in \mathrm{Hom}(M, N)$ and $\alpha \in \mathbb{K}$ we have $F\alpha : M \ni v \mapsto (-1)^{p(v)p(\alpha)} F(v)\alpha \in N$.

**Notation 2.1.** *We denote by $M^*$ the $\mathbb{K}$-module* $\mathrm{Hom}(M, \mathbb{K})$.

**Definition 2.3.** *We say that $A$ is a $\mathbb{K}$-superalgebra if it is a $\mathbb{K}$-module equipped with a distributive application $A \times A \xrightarrow{\cdot} A$ such that $p(a \cdot b) = p(a) + p(b)$ and $(a \cdot b)\alpha = a \cdot (b\alpha) = (-1)^{p(b)p(\alpha)}(a\alpha) \cdot b$, for any $a, b \in A$ and $\alpha \in \mathbb{K}$.*

*We say that the $\mathbb{K}$-superalgebra $A$ is commutative if $a \cdot b = (-1)^{p(a)p(b)} b \cdot a$ for $a, b \in A$; and $c^2 = 0$ for any $c \in A_1$.*

Let $M, N$ be two $\mathbb{K}$-modules. We denote by $M \otimes N$ the $\mathbb{K}$-module generated by $\{v \otimes w;\ v \in M, w \in N\}$ with relations

$$\begin{aligned}
&(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \\
&v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2 \\
&(v \otimes w)\alpha = v \otimes w\alpha = (-1)^{p(w)p(\alpha)} v\alpha \otimes w \qquad \forall \alpha \in \mathbb{K}
\end{aligned}$$

and graduation $p(v \otimes w) = p(v) + p(w)$.

The *tensor algebra* of $M$ is defined as $T(M) := \mathbb{K} + M + (M \otimes M) + (M \otimes M \otimes M) + \cdots$, with product $(v_1 \otimes \cdots \otimes v_i) \cdot (v_{i+1} \otimes \cdots \otimes v_n) = v_1 \otimes \cdots \otimes v_n$, for all $i, n \in \mathbb{N}$. It is an associative $\mathbb{K}$-superalgebra.

The *symmetric algebra* $S(M)$ of $M$ is defined as the quotient $S(M) := T(M)/I$, where $I$ is the ideal generated by

$$\{v \otimes w - (-1)^{p(v)p(w)} w \otimes v,\ u \otimes u | v, w \in M, u \in M_1\}.$$

It is a commutative and associative $\mathbb{K}$-superalgebra. Moreover, it has the decomposition $S(M) = \mathbb{K} \oplus \bigoplus_{n=1}^{\infty} S^n(M)$, where $S^n(M)$ is the $\mathbb{K}$-module generated by products of $n$ elements of $M$.

**Definition 2.4.** *Let $A$ and $B$ be two $\mathbb{K}$-superalgebras. An even map $F \in \mathrm{Hom}(A, B)_0$ is said to be a morphism of $\mathbb{K}$-superalgebras if $F(ab) = F(a)F(b)$ for any $a, b \in A$.*

Let $n \geqslant 2$, and $\gamma \in \mathrm{Hom}(\overbrace{M \otimes \cdots \otimes M}^{n}, N)$. It is called a *multilinear form (or $n$-linear form)* on $M$ with values in $N$. We write $\gamma(a_1 \otimes \cdots \otimes a_n) = \gamma(a_1, \ldots, a_n)$ for any $a_1, \ldots, a_n \in M$, identifying $\gamma$ with a multilinear function on $\underbrace{M \times \cdots \times M}_{n}$ with values in $N$.

**Definition 2.5.** *We say that $\gamma$ is symmetric if*

$$\gamma(a_1, \ldots, a_i, a_{i+1}, \ldots, a_n) = (-1)^{p(a_i)p(a_{i+1})}\gamma(a_1, \ldots, a_{i+1}, a_i, \ldots, a_n) \qquad \text{for any} \quad i = 1, \ldots, n-1,$$

*and*

$$\gamma(a_1, \ldots, a_i, a_i, \ldots, a_n) = 0 \qquad \text{if } p(a_i) = 1 \qquad \text{for any} \quad i = 1, \ldots, n.$$

Since $S^n(M)$ is a quotient of $\overbrace{M \otimes \cdots \otimes M}^{n}$, this identifies $\mathrm{Hom}(S^n(M), N)$ with the space of symmetric multilinear functions on $\underbrace{M \times \cdots \times M}_{n}$ with values in $N$.

### *2.2. Formal functions and extensions of scalars*

The following examples and notation are fundamental in this text.

Let $M$ be a $\mathbb{K}$-module. We introduce the morphism of $\mathbb{K}$-superalgebras $\Delta_M : S(M) \to S(M) \otimes S(M)$ such that $M \ni v \mapsto v \otimes 1 + 1 \otimes v$; it is called the natural coproduct of $S(M)$. It is a *commutative coproduct*: if $w \in S(M)$ and $\Delta_M(w) = \sum_i w_i \otimes w_i'$, we have $\sum_i w_i \otimes w_i' = \sum_i (-1)^{p(w_i')p(w_i)} w_i' \otimes w_i$. Moreover, it is *associative*:

$$(id_{S(M)} \otimes \Delta_M) \circ \Delta_M = (\Delta_M \otimes id_{S(M)}) \circ \Delta_M.$$

Let $p \in \mathbb{N}$ and $v_1, \ldots, v_p \in M$. For any permutation $(i_1, \ldots, i_p)$ of $\{1, \ldots, p\}$ we denote by $\mathrm{sig}(v_{i_1}, \ldots, v_{i_p}) \in \{1, -1\}$ the sign such that

$$\mathrm{sig}(v_{i_1}, \ldots, v_{i_p}) v_{i_1} \cdots v_{i_p} = v_1 \cdots v_p$$

in $S(M)$. We have

$$\Delta_M(v_1 \cdots v_p) = \sum_{j=0}^{p} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant p} \sigma(v, \underline{i}) v_{i_1} \cdots v_{i_j} \otimes v_1 \cdots \widehat{v_{i_1}} \cdots \widehat{v_{i_j}} \cdots v_p$$

where the symbols exhibiting a superimposed ‘hat’ are omitted, and $\sigma(v, \underline{i}) := \mathrm{sig}(v_{i_1}, \ldots, v_{i_j}, v_1, \ldots, \widehat{v_{i_1}}, \ldots, \widehat{v_{i_j}}, \ldots, v_p)$.

**Example 2.1.** Let $N$ be a $\mathbb{K}$-module. In $\mathrm{Hom}(S(M), N)$ the coproduct $\Delta_M$ allows us to define the multiplication by an element of $S(M)^*$ by the formula

$$F\varphi = (F \otimes \varphi) \circ \Delta_M \qquad \text{for} \quad F \in \mathrm{Hom}(S(M), N), \quad \varphi \in S(M)^*.$$

In particular, $S(M)^*$ is an associative $\mathbb{K}$-superalgebra, called the algebra of formal functions on $M$, and $\mathrm{Hom}(S(M), N)$ is a $S(M)^*$-module, called the space of *formal functions* on $M$ with values in $N$.

When $N = M$, $F \in \mathrm{Hom}(S(M), M)$ is called a *formal vector field on M*.

**Remark 2.1.** With $w \in N$ we associate the morphism of $\mathbb{K}$-modules $S(M) \to N$ such that $1 \mapsto w$, $S^n(M) \mapsto \{0\}$ if $n \neq 0$. We consider it as the constant map on $M$ with value $w$. We still denote this map by $w$. This provides a natural injection $N \subseteq \mathrm{Hom}(S(M), N)$.

**Remark 2.2.** By the previous remark, there is a natural morphism of $S(M)^*$-modules of $S(M)^* \otimes N$ in $\mathrm{Hom}(S(M), N)$, which is in general not injective and not surjective. The image could be called the space of polynomial functions on $M$ with values in $N$.

Traditionally, the *extension of scalars* from $\mathbb{K}$ to $S(M)^*$ for $N$ is the space $S(M)^* \otimes N$.

In this paper, the space $\mathrm{Hom}(S(M), N)$ is more useful, and we consider it also as an extension of scalars from $\mathbb{K}$ to $S(M)^*$. In practice, a $\mathbb{K}$-linear statement about $\mathbb{K}$-modules $N_1, N_2, \ldots$ will extend to a $S(M)^*$-linear statement about $S(M)^* \otimes N_1$, $S(M)^* \otimes N_2, \ldots$, $\mathrm{Hom}(S(M), N_1)$, $\mathrm{Hom}(S(M), N_2), \ldots,$ with natural $S(M)^*$-morphisms $S(M)^* \otimes N_j \to \mathrm{Hom}(S(M), N_j)$.

**Example 2.2.** Let $N_1, N_2, N, M$ be $\mathbb{K}$-modules, and $\phi : N_1 \otimes N_2 \to N$ a bilinear map. For $F \in \mathrm{Hom}(S(M), N_1)$, $G \in \mathrm{Hom}(S(M), N_2)$, we define

$$\phi(F, G) = \phi \circ (F \otimes G) \circ \Delta_M.$$

Then $\phi$ is a $S(M)^*$-bilinear map from $\mathrm{Hom}(S(M), N_1) \otimes \mathrm{Hom}(S(M), N_2)$ to $\mathrm{Hom}(S(M), N)$.

There is a similar definition with any finite number of modules $N_1, \ldots, N_n$, $n \geqslant 2$.

Similarly, if $\phi = \phi(a_1, \ldots, a_n)$ is a $\mathbb{K}$-multilinear form on $N$ with values in $\mathbb{K}$, then $\phi$ extends in a natural way to a $S(M)^*$-multilinear form $\underbrace{\mathrm{Hom}(S(M), N) \otimes \cdots \otimes \mathrm{Hom}(S(M), N)}_{n} \to S(M)^*$.

**Remark 2.3.** In particular, if $N$ has a $\mathbb{K}$-superalgebra structure given by $\phi \in \mathrm{Hom}(N \otimes N, N)$, then $\mathrm{Hom}(S(M), N)$ is a $S(M)^*$-superalgebra.

**Definition 2.6.** *Let N be a* $\mathbb{K}$*-module. For any formal vector field* $\varphi \in \mathrm{Hom}(S(M), M)$ *and* $F \in \mathrm{Hom}(S(M), N)$*, we introduce*

$$L(\varphi)(F) := (-1)^{p(\varphi)p(F)} F \circ \mathrm{Mult}_{S(M)} \circ (id_{S(M)} \otimes \varphi) \circ \Delta_M. \tag{4}$$

*It belongs to* $\mathrm{Hom}(S(M), N)$*, and it is called the derivative of F in the direction of* $\varphi$*. If N is a superalgebra* $L(\varphi)$ *satisfies the Leibniz rule.*

**Remark 2.4.** Let $\varphi, \psi \in \mathrm{Hom}(S(M), M)$. We have the usual formula for the bracket of derivatives in the direction of vector fields:

$$L(\varphi) \circ L(\psi) - (-1)^{p(\psi)p(\varphi)} L(\psi) \circ L(\varphi) = L\left(L(\varphi)(\psi) - (-1)^{p(\psi)p(\varphi)} L(\psi)(\varphi)\right).$$

*2.3. Formal differential forms over a module*

Let $M$ be a $\mathbb{K}$-module. To introduce the notion of differential form over $M$ we need the following definition.

**Definition 2.7.** *We denote by $\Pi M$ the $\mathbb{K}$-module with graduation $(\Pi M)_0 = M_1$ and $(\Pi M)_1 = M_0$. The identity over $M$ gives an odd map $\pi \in \mathrm{Hom}(M, \Pi M)_1$. The structure of the $\mathbb{K}$-module for $\Pi M$ is given by*

$$(\pi m)\alpha := \pi(m\alpha) \qquad \forall m \in M, \quad \forall \alpha \in \mathbb{K}.$$

Let $N$ be a $\mathbb{K}$-module.

**Definition 2.8.** *Let $n \in \mathbb{N}$. A formal differential $n$-form over $M$ and with values in $N$ is an element of*

$$A^n(M, N) := \mathrm{Hom}(S^n(\Pi M) \otimes S(M), N).$$

*The space of formal differential forms is $A(M, N) := \bigoplus_{n=0}^{\infty} A^n(M, N)$.*

We write $A(M) = \bigoplus_{n=0}^{\infty} A^n(M)$ for $A(M, \mathbb{K})$. Note that $A^0(M, N) = \mathrm{Hom}(S(M), N)$ is the space of formal functions on $M$ with values in $N$.

From now on we will omit the word 'formal' and we will speak of differential forms.

**Remark 2.5.** $S(\Pi M \oplus M) = S(\Pi M) \otimes S(M)$.

So, we could also consider the larger space of (non-graded) differential forms $\mathrm{Hom}(S(\Pi M \oplus M), N)$. Clearly $A(M)$ is a subalgebra of $S(\Pi M \oplus M)^*$, graded by the spaces $A^n(M)$: if $\alpha_n \in A^n(M)$ and $\beta_p \in A^p(M)$, we have $\alpha_n\beta_p \in A^{n+p}(M)$. In the same way, $A(M, N)$ is a graded $A(M)$-module.

**Example 2.3.** Let $n = p = 1$. For any $w \in S(M)$ we use the notation $\Delta_M(w) = \sum_i w_i \otimes w_i'$. The differential 2-form $\alpha_1\beta_1$ verifies

$$\begin{aligned}\alpha_1\beta_1(\pi m\pi n \otimes w) &= \sum_i \alpha_1(\pi m \otimes w_i)\beta_1(\pi n \otimes w_i')(-1)^{p(\beta_1)p(\pi m + w_i)+p(\pi n)p(w_i)} \\ &\quad + \sum_i (-1)^{p(\pi m)p(\pi n)}\alpha_1(\pi n \otimes w_i)\beta_1(\pi m \otimes w_i')(-1)^{p(\beta_1)p(\pi n + w_i)+p(\pi m)p(w_i)}\end{aligned}$$

for any $m, n \in M$.

If $\beta \in A(M, \Pi M \oplus M)$ is a (graded) vector field on $\Pi M \oplus M$, then the derivative $L(\beta)$ (see definition 2.6) acts in $A(M, N)$. For example, $L(\varphi)(\alpha)$ is defined for any $\alpha \in A(M, N)$ and $\varphi \in \mathrm{Hom}(S(M), M)$, and $L(\varphi)(A^n(M, N)) \subset A^n(M, N)$. In particular, if $n \in \mathbb{N}$, $a \in M$ and $\alpha_n \in A^n(M, N)$, $L(a)(\alpha_n)$ is the differential $n$-form such that, for any $\pi m_1 \cdots \pi m_n \in S^n(\Pi M)$ and $w \in S(M)$, we have

$$L(a)(\alpha_n)(\pi m_1 \cdots \pi m_n \otimes w) = (-1)^{p(a)(p(\alpha_n)+p(\pi m_1+\cdots+\pi m_n))}\alpha_n(\pi m_1 \cdots \pi m_n \otimes aw).$$

It is called the *Lie derivative* of $\alpha_n$ in the direction $a$. We also introduce

$$i(\varphi)(\alpha) := L(\pi \circ \varphi)(\alpha).$$

It is called the *contraction* of $\alpha$ with $\varphi$. Let $n \geqslant 1, \alpha_n \in A^n(M, N)$. We have $i(\varphi)(\alpha_n) \in A^{n-1}(M, N)$. In particular, if $a \in M$, for any $\pi m_1, \ldots, \pi m_n \in \Pi M$ we have

$$i(a)(\alpha)(\pi m_1 \cdots \pi m_{n-1} \otimes \cdot) = (-1)^{p(\pi a)p(\alpha)}\alpha(\pi a\pi m_1 \cdots \pi m_n \otimes \cdot).$$

By induction we get that

$$i(a_1)\cdots i(a_n)(\alpha_n) = (-1)^{p(\alpha_n)p(\pi a_1+\cdots+\pi a_n)}\alpha_n(\pi a_1\cdots\pi a_n\otimes\cdot) \in \mathrm{Hom}(S(M),N) \tag{5}$$

for all $a_1,\ldots,a_n \in \mathfrak{g}$.

To build differential forms, we will use sometimes the following lemma.

Let $n \geqslant 2$ and $\gamma : \overbrace{M\otimes\cdots\otimes M}^{n} \to N$ be a multilinear form.

**Definition 2.9.** *We say that $\gamma$ is antisymmetric if*

$$\gamma(a_1,\ldots,a_i,a_{i+1},\ldots,a_n) = -(-1)^{p(a_i)p(a_{i+1})}\gamma(a_1,\ldots,a_{i+1},a_i,\ldots,a_n)$$
$$\textit{for any}\quad i=1,\ldots,n-1$$

*and*

$$\gamma(a_1,\ldots,a_i,a_i,\ldots,a_n)=0 \qquad \textit{if}\quad p(a_i)=0, \qquad \textit{for any}\quad i=1,\ldots,n.$$

**Lemma 2.1.** *If $\gamma$ is antisymmetric, there exists a unique $\tilde\gamma \in \mathrm{Hom}(S^n(\Pi M),N)$ such that*

$$\tilde\gamma(\pi a_1\cdots\pi a_n) = (-1)^{p(a_{n-1})+p(a_{n-3})+\cdots}\gamma(a_1,\ldots,a_n)$$

*for any $a_1,\ldots,a_n\in\mathfrak{g}$. The degree of $\tilde\gamma$ is given by $p(\tilde\gamma)=p(\gamma)+n$ modulo 2.*

**Remark 2.6.** Replacing $N$ by $\mathrm{Hom}(S(M),N)$, we obtain the formula

$$\tilde\gamma(\pi a_1\cdots\pi a_n\otimes\cdot) = (-1)^{p(a_{n-1})+p(a_{n-3})+\cdots}\gamma(a_1,\cdots,a_n)(\cdot) \tag{6}$$

which identifies $A^n(M,N)$ with the space of antisymmetric multilinear functions on $\overbrace{M\times\cdots\times M}^{n}$ with values in $\mathrm{Hom}(S(M),N)$.

Let $n\geqslant 2$, $\alpha\in A^n(M,N)$, $a_1,\ldots,a_n\in M$. We use the notation $i(a_1)\cdots i(a_n)(\alpha)$ and $\alpha(\pi a_1,\ldots,\pi a_n)$. They are related (see formula (5)) by

$$i(a_1)\cdots i(a_n)(\alpha) = (-1)^{p(\alpha)p(\pi a_1+\cdots+\pi a_n)}\alpha(\pi a_1,\ldots,\pi a_n).$$

We introduce the map $\delta\in\mathrm{Hom}(S(M\oplus\Pi M),M\oplus\Pi M)$ such that

$$\delta(\pi m)=m \qquad \forall m\in M$$
$$\delta(M)=\{0\}$$
$$\delta(S^n(M\oplus\Pi M))=\{0\} \qquad \forall n\neq 1.$$

The *de Rham differential* of $\alpha\in A(M,N)$ is the differential form

$$\mathrm{d}\alpha := -L(\delta)(\alpha).$$

Let $n\geqslant 1$ and $\alpha_{n-1}\in A^{n-1}(M,N)$. Its de Rham differential is the differential $n$-form such that, for any $m_1,\ldots,m_n\in M$ and $w\in S(M)$, we have

$$\mathrm{d}\alpha_{n-1}(\pi m_1\cdots\pi m_n\otimes w)$$
$$= (-1)^{p(\alpha_{n-1})}\sum_{i=1}^{n}(-1)^i\alpha_{n-1}(\pi m_1\cdots\widehat{\pi m_i}\cdots\pi m_n\otimes m_i\cdot w)\mathrm{sig}(\vec m,i)$$

where $\mathrm{sig}(\vec m,i) := (-1)^{p(m_1+\cdots+m_{i-1})+p(m_i)p(\pi m_{i+1}+\cdots+\pi m_n)}$.

Using remark 2.4 we get the following usual Cartan commutation rules:

$$\mathrm{d} \circ \mathrm{d} = 0$$
$$i(\varphi) \circ L(\psi) - (-1)^{p(\pi\varphi)p(\psi)} L(\psi) \circ i(\varphi) = -(-1)^{p(\varphi)p(\psi)} i(L(\psi)(\varphi)) \qquad \forall \varphi, \psi \in \mathrm{Hom}(S(M), M)$$
$$i(\varphi) \circ i(\psi) = (-1)^{p(\pi\varphi)p(\pi\psi)} i(\psi) \circ i(\varphi)$$
$$\mathrm{d} \circ i(a) - (-1)^{p(\pi a)} i(a) \circ \mathrm{d} = L(a) \qquad \forall a \in M$$
$$\mathrm{d} \circ L(a) = (-1)^{p(\pi a)} L(a) \circ \mathrm{d}.$$

We call $A(M, N)$ the *formal de Rham complex* on $M$ with values in $N$.

**Remark 2.7.** Let $\alpha_0 \in A^0(M, N)$, $\alpha_1 \in A^1(M, N)$, and $\alpha_2 \in A^2(M, N)$. For any $a, b, c \in M$ we have

$$i(a)(\mathrm{d}\alpha_0) = (-1)^{p(a)} L(a)(\alpha_0)$$
$$i(a)i(b)(\mathrm{d}\alpha_1) = -(-1)^{p(a)+p(b)} L(a)(i(b)(\alpha_1)) + (-1)^{p(a)p(b)} L(b)(i(a)(\alpha_1))$$
$$i(a)i(b)i(c)(\mathrm{d}\alpha_2) = (-1)^{p(a+b+c)} L(a)(i(b)i(c)(\alpha_2)) + (-1)^{p(c)+p(c)(p(a)+p(b))} \times L(c)(i(a)i(b)(\alpha_2)) - (-1)^{p(a)p(b)+p(c)} L(b)(i(a)i(c)(\alpha_2)).$$

## 3. The generic point of a Lie superalgebra

From now on we assume that $\frac{1}{2} \in \mathbb{K}_0$.

We recall the definition of a Lie superalgebra.

**Definition 3.1.** *Let $\mathfrak{g}$ be a $\mathbb{K}$-superalgebra such that its product $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ verifies*

$$[X, Y] = -(-1)^{p(X)p(Y)}[Y, X] \qquad \forall X, Y \in \mathfrak{g} \tag{7}$$
$$[X, X] = 0 \qquad \forall X \in \mathfrak{g}_0 \tag{8}$$
$$[[X, Y], Z] = [X, [Y, Z]] - (-1)^{p(Y)p(X)}[Y, [X, Z]] \qquad \forall X, Y, Z \in \mathfrak{g} \tag{9}$$
$$[Y, [Y, Y]] = 0 \qquad \forall Y \in \mathfrak{g}_1. \tag{10}$$

*Such a $\mathfrak{g}$ is called a Lie $\mathbb{K}$-superalgebra.*

**Remark 3.1.** Since $2 \in \mathbb{K}$ is invertible (8) follows from (7). If $3 \in \mathbb{K}$ is invertible (10) follows from (7) and (9).

When $\mathfrak{g} = \mathfrak{g}_0$ and $\mathbb{K} = \mathbb{K}_0 \supseteq \mathbb{Q}$ is a field we have an *ordinary Lie algebra*. As explained in [3], if $\mathfrak{g}_1 \neq \{0\}$ and $2 \in \mathbb{K}$ is not invertible, definition 3.1 is not the right one, and we avoid this problem.

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. This section is devoted to introducing some preliminary properties of $\mathfrak{g}_x := \mathrm{Hom}(S(\mathfrak{g}), \mathfrak{g})$. For the proofs or for more details see section 3 of [10]. We recall that $\mathfrak{g} \subseteq \mathfrak{g}_x$ (see notation in remark 2.1), and that $\mathfrak{g}_x$ has a structure of $S(\mathfrak{g})^*$-superalgebra (see example 2.2).

**Lemma 3.1.** *$\mathfrak{g}_x$ is a Lie $S(\mathfrak{g})^*$-superalgebra, and $\mathfrak{g} \subseteq \mathfrak{g}_x$ is a Lie $\mathbb{K}$-subsuperalgebra. Similarly, $A(\mathfrak{g}, \mathfrak{g})$ is a (graded) Lie $A(\mathfrak{g})$-superalgebra.*

**Remark 3.2.** Let $M$ be any $\mathbb{K}$-module. We remark that $\mathrm{Hom}(S(M), M)$ is in duality with the derivations of $S(M)^*$. This gives a Lie superalgebra structure on $\mathrm{Hom}(S(M), M)$ (coming from the bracket of the corresponding derivations). It is $\mathbb{K}$-linear, but not $S(M)^*$-linear (here we do not need $\frac{1}{2} \in \mathbb{K}_0$).

In the particular case $M = \mathfrak{g}$, this structure is different from the Lie superalgebra structure in lemma 3.1—which comes from the Lie superalgebra structure on $\mathfrak{g}$.

The identity over $\mathfrak{g}$ is called the *generic point* of $\mathfrak{g}$. It extends to the morphism of $\mathbb{K}$-modules $x : S(\mathfrak{g}) \to \mathfrak{g}$ such that $S^n(\mathfrak{g}) \mapsto \{0\}$ if $1 \neq n$[1]. In particular, we can consider $[x, a] \in \mathfrak{g}_x$ for any $a \in \mathfrak{g}$. We use the standard notation

$$(\operatorname{ad} x)(a) := [x, a].$$

For any $n \in \mathbb{N}$, we identify $\operatorname{Hom}(S^n(\mathfrak{g}), \mathfrak{g})$ with the subspace of $F \in \operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$ which is zero on $S^m(\mathfrak{g})$, $m \neq n$.

**Remark 3.3.** For any $n \in \mathbb{N}$, we have $(\operatorname{ad} x)^n(a) \in \operatorname{Hom}(S^n(\mathfrak{g}), \mathfrak{g})$.

We denote by $\mathbb{K}_0[[t]]$ the ring of formal power series in $t$, with coefficients in $\mathbb{K}_0$; and by $\mathbb{K}_0[[t, u]]$ the ring of formal power series in variables $t, u$ and coefficients in $\mathbb{K}_0$.

Let $a \in \mathfrak{g}$ and $q(t) = \sum_{k=0}^{\infty} c_k t^k \in \mathbb{K}_0[[t]]$. We denote by

$$q^a := q(\operatorname{ad} x)(a) \equiv c_0 a + c_1 [x, a] + \cdots \in \mathfrak{g}_x \tag{11}$$

the formal vector field such that its restriction to $S^n(\mathfrak{g})$ is $c_n (\operatorname{ad} x)^n(a)$, for any $n \geqslant 0$.

Let $p, q \in \mathbb{N}$ and $a, b \in \mathfrak{g}$. We introduce the notation

$$(t^p u^q : [a, b])_x := [(\operatorname{ad} x)^p(a), (\operatorname{ad} x)^q(b)] \in \mathfrak{g}_x$$

which is extended to any formal power series in $t$ and $u$.

Let $b \in \mathfrak{g}$. We recall that the notation $L(b)$ has been introduced in (4).

**Theorem 3.1** ([10], theorem 3.2)**.** *For any $a \in \mathfrak{g}$ and $q \in \mathbb{K}_0[[t]]$ we have*

$$L(b)(q^a) = \left( \frac{q(t+u) - q(u)}{t} : [b, a] \right)_x .$$

**Theorem 3.2** (see [10], lemma 4.3)**.** *Let $\omega(t, u) \in \mathbb{K}_0[[t, u]]$. If for any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ we have*

$$(\omega(t, u), [a, b])_x = 0 \qquad \forall a, b \in \mathfrak{g}$$

*then $\omega(t, u) = 0$.*

## 4. Universal differential forms on a Lie superalgebra

Let $q(t) = \sum_{k=0}^{\infty} c_k t^k \in \mathbb{K}_0[[t]]$. With any Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$ we associate an odd, $\mathfrak{g}$-valued, differential 1-form $\alpha_{\mathfrak{g}} \in \operatorname{Hom}(\Pi\mathfrak{g} \otimes S(\mathfrak{g}), \mathfrak{g})_1 = A^1(\mathfrak{g}, \mathfrak{g})_1$ by the formula

$$i(a)(\alpha_{\mathfrak{g}}) = (-1)^{p(a)} q^a \qquad \text{for } a \in \mathfrak{g}.$$

For example, for $a, u, v \in \mathfrak{g}$ we have

$$\begin{aligned} &\alpha_{\mathfrak{g}}(\pi a) = -c_0 a \\ &\alpha_{\mathfrak{g}}(\pi a \otimes u) = -(-1)^{p(a)p(u)} c_1 [u, a] \\ &\alpha_{\mathfrak{g}}(\pi a \otimes uv) = (-1)^{p(a)(p(u)+p(v))} c_2([u, [v, a]] + (-1)^{p(u)p(v)} [v, [u, a]]). \end{aligned} \tag{12}$$

[1] This definition is valid if $\mathfrak{g}$ is replaced by any $\mathbb{K}$-module.

The differential form $\alpha_{\mathfrak{g}}$ is functorial in $\mathfrak{g}$ in an obvious sense. This is why we call such a form *universal*. In fact, as in section 7 of [10], it can be proved that, in the category of Lie algebras over a commutative $\mathbb{Q}$-algebra, any universal differential form is obtained in this manner.

We shall need the following formulae, valid for all $\mathfrak{g}$.

**Lemma 4.1.** *For any $a, b \in \mathfrak{g}$ we have*

$$i(a)i(b)(\mathrm{d}\alpha_{\mathfrak{g}}) = (-1)^{p(a)+1}\left(\frac{q(u+t)-q(u)}{t} + \frac{q(u+t)-q(t)}{u} : [a,b]\right)_x \tag{13}$$

$$\tfrac{1}{2}i(a)i(b)([\alpha_{\mathfrak{g}},\alpha_{\mathfrak{g}}]) = (-1)^{p(a)+1}(q(t)q(u) : [a,b])_x. \tag{14}$$

**Proof.** We consider the first formula. Using remark 2.7 we see that

$$i(a)i(b)(\mathrm{d}\alpha_{\mathfrak{g}}) = -(-1)^{p(a)}L(a)(q^b) + (-1)^{p(a)p(b)+p(a)}L(b)(q^a).$$

Formula (13) follows from theorem 3.1 and from the identity

$$\left(\frac{q(u+t)-q(u)}{t} : [b,a]\right)_x = -(-1)^{p(a)p(b)}\left(\frac{q(u+t)-q(t)}{u} : [a,b]\right)_x.$$

Now we consider formula (14). Since $i(b)$ is a derivation, we have $i(b)([\alpha_{\mathfrak{g}},\alpha_{\mathfrak{g}}]) = [i(b)(\alpha_{\mathfrak{g}}),\alpha_{\mathfrak{g}}] + (-1)^{p(b)+1}[\alpha_{\mathfrak{g}}, i(b)(\alpha_{\mathfrak{g}})]$. Since $i(a)$ is a derivation, and since $i(a)i(b)(\alpha_{\mathfrak{g}}) = 0$, we obtain $i(a)i(b)([\alpha_{\mathfrak{g}},\alpha_{\mathfrak{g}}]) = 2(-1)^{p(b)+1}[i(a)(\alpha_{\mathfrak{g}}), i(b)(\alpha_{\mathfrak{g}})] = 2(-1)^{p(a)+1}[q^a, q^b]$. □

## 5. Maurer–Cartan equations

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. We call a *left invariant (formal) Maurer–Cartan form* over $\mathfrak{g}$, each differential odd 1-form $\bar{\alpha} \in \mathrm{Hom}(\Pi\mathfrak{g} \otimes S(\mathfrak{g}), \mathfrak{g})_1 = A^1(\mathfrak{g},\mathfrak{g})_1$ such that,

(i) for any $a \in \mathfrak{g}$,

$$(-1)^{p(a)}i(a)(\bar{\alpha})(1) \equiv -\bar{\alpha}(\pi a) = a$$

(ii) its de Rham differential $\mathrm{d}\bar{\alpha} \in A^2(\mathfrak{g},\mathfrak{g})_0$ verifies the Maurer–Cartan equation

$$\mathrm{d}\bar{\alpha} = -\tfrac{1}{2}[\bar{\alpha},\bar{\alpha}].$$

In an analogous way we have a notion of *right invariant (formal) Maurer–Cartan form* over $\mathfrak{g}$. We denote by $\tilde{\alpha}$ such a differential 1-form. The difference with $\bar{\alpha}$ is that the de Rham differential $\mathrm{d}\tilde{\alpha}$ verifies another form of the equation of Maurer–Cartan:

$$\mathrm{d}\tilde{\alpha} = \tfrac{1}{2}[\tilde{\alpha},\tilde{\alpha}]. \tag{15}$$

**Remark 5.1.** Let $\mathbb{K} = \mathbb{R}$ and $\mathfrak{g} = \mathfrak{g}_0$ be a finite-dimensional Lie algebra. If $G$ is a local Lie group with Lie algebra $\mathfrak{g}$, the left and right invariant Maurer–Cartan forms on $G$ satisfy the Maurer–Cartan equations. Pulling back these forms on $\mathfrak{g}$ by a local analytic diffeomorphism from $\mathfrak{g}$ to $G$, we get analytic Maurer–Cartan forms defined in a neighbourhood of $0 \in \mathfrak{g}$. Considering their Taylor expansion at the origin, we get formal Maurer–Cartan forms on $\mathfrak{g}$.

Conversely, an analytic Maurer–Cartan form defined in a neighbourhood of $0 \in \mathfrak{g}$ can be used to prove the third Lie theorem. It states the existence of a local analytic Lie group $G$ with Lie algebra $\mathfrak{g}$, and of a well-defined local analytic diffeomorphism from $\mathfrak{g}$ to $G$ such that the pull back of the left invariant Maurer–Cartan form on $G$ is the given Maurer–Cartan form on $\mathfrak{g}$ (see, for example, [11], from which we borrow the notation $\bar{\alpha}$ and $\tilde{\alpha}$).

Theorems 5.1 and 5.2 provide analytic Maurer–Cartan forms on $\mathfrak{g}$—the ones which are obtained from the Maurer–Cartan forms on $G$ using the exponential map. We think that our theorem can be used to simplify the classical exposition of the third Lie theorem and of the existence of the exponential map.

We think that it is an interesting fact—and, as far as we know, this idea appeared for the first time in [9]—that the well-known formulae for the derivative of the exponential map from $\mathfrak{g}$ to $G$ are related to the functional equations (16) and (17).

We are looking for Maurer–Cartan forms, universal in the sense of section 4. This means that we are looking for formal series $\bar{f}$, $\tilde{f} \in \mathbb{K}_0[[t]]$ such that the corresponding forms, defined by the formulae

$$(-1)^{p(a)} i(a)(\bar{\alpha}_{\mathfrak{g}}) = \bar{f}(\operatorname{ad} x)(a)$$
$$(-1)^{p(a)} i(a)(\tilde{\alpha}_{\mathfrak{g}}) = \tilde{f}(\operatorname{ad} x)(a)$$

are (respectively left and right invariant) Maurer–Cartan forms on $\mathfrak{g}$, for all Lie $\mathbb{K}$-superalgebra $\mathfrak{g}$.

**Theorem 5.1.** *Let $\bar{f} \in \mathbb{K}_0[[t]]$. Then $\bar{\alpha}_{\mathfrak{g}}$ is a left invariant Maurer–Cartan form for all Lie $\mathbb{K}$-superalgebra if and only if*

$$\begin{cases} \bar{f}(0) = 1 \\ \dfrac{\bar{f}(u+t) - \bar{f}(u)}{t} + \dfrac{\bar{f}(u+t) - \bar{f}(t)}{u} + \bar{f}(u)\bar{f}(t) = 0 \end{cases} \tag{16}$$

*in $\mathbb{K}_0[[t, u]]$.*

*Let $\tilde{f} \in \mathbb{K}_0[[t]]$. Then $\tilde{\alpha}_{\mathfrak{g}}$ is a right invariant Maurer–Cartan form for all Lie $\mathbb{K}$-superalgebra if and only if*

$$\begin{cases} \tilde{f}(0) = 1 \\ \dfrac{\tilde{f}(u+t) - \tilde{f}(u)}{t} + \dfrac{\tilde{f}(u+t) - \tilde{f}(t)}{u} - \tilde{f}(u)\tilde{f}(t) = 0 \end{cases} \tag{17}$$

*in $\mathbb{K}_0[[t, u]]$.*

**Proof.** The fact that (16) and (17) are sufficient follows from (12), (13) and (14). The converse follows from theorem 3.2. □

**Theorem 5.2.** *Equations (16) have a solution in $\mathbb{K}_0[[t]]$ if and only if $\mathbb{K} \supseteq \mathbb{Q}$. In this case there exists a unique solution, given by*

$$\bar{f}(t) := \frac{1 - \mathrm{e}^{-t}}{t}.$$

*Equations (17) have a solution in $\mathbb{K}_0[[t]]$ if and only if $\mathbb{K} \supseteq \mathbb{Q}$. In this case there exists a unique solution, given by*

$$\tilde{f}(t) := \frac{\mathrm{e}^{t} - 1}{t}.$$

**Proof.** Let us prove the assertion for $\tilde{f}$, the case of $\bar{f}$ being similar. Assume that there exists a solution $f = 1 + \sum_{k=1}^{\infty} c_k t^k$ of equations (17). Evaluating at $u = 0$, we obtain $\frac{f(t)-1}{t} + f'(t) = f(t)$. This gives

$$2c_1 = 1, \qquad (k+1)c_k = c_{k-1} \qquad \forall k \geqslant 2.$$

We get $c_1 = \frac{1}{2}$. By induction we see that $k+1$ is invertible in $\mathbb{K}$ for all $k \geqslant 2$, and that $c_k = \frac{1}{(k+1)!}$. Thus $\mathbb{K} \supseteq \mathbb{Q}$ and $f(t) = \frac{e^t - 1}{t}$.

Conversely, let us suppose that $\mathbb{K} \supseteq \mathbb{Q}$. Then it is easy to verify that $\tilde{f}(t) = \frac{e^t - 1}{t}$ is a solution of (17). □

**Remark 5.2.** Let $N \geqslant 2$, and $\mathfrak{g}$ be an $N$-nilpotent Lie superalgebra, that is $\operatorname{ad} X_1 \circ \cdots \circ \operatorname{ad} X_N = 0$ for all $X_1, \ldots, X_N \in \mathfrak{g}$. Then, assuming that $1, 2, 3, \ldots, N$ are invertible in $\mathbb{K}_0$, the truncated series $\bar{f}$ modulo $t^N$ and $\tilde{f}$ modulo $t^N$ provide Maurer–Cartan forms on $\mathfrak{g}$.

## 6. Quadratic Lie superalgebras

### *6.1. Definitions*

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra. We say that a bilinear form $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$ is *invariant* if $\gamma(X, [Y, Z]) = \gamma([X, Y], Z)$ for any $X, Y, Z \in \mathfrak{g}$.

**Lemma 6.1.** *Let $\mathfrak{g}$ be equipped with an invariant bilinear form $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$. For all $X \in \mathfrak{g}_0$ and $Y, Z \in \mathfrak{g}$ we have*

$$\gamma\big((\operatorname{ad} X)^j(Y), Z\big) = \gamma\big(Y, (-\operatorname{ad} X)^j(Z)\big) \qquad \forall j \in \mathbb{N}.$$

**Proof.** The statement follows by induction. □

We say that $\gamma$ is *non-degenerate* if $\mathfrak{g} \ni X \mapsto \gamma(X, \cdot) \in \operatorname{Hom}(\mathfrak{g}, \mathbb{K})$ is one-to-one.

**Definition 6.1.** *Let $\mathbb{K} = \mathbb{K}_0$ be a field, $\mathfrak{g}_0$ and $\mathfrak{g}_1$ be finite-dimensional $\mathbb{K}$-supervector spaces. If $\gamma$ is even, symmetric, invariant and non-degenerate, we say that $(\mathfrak{g}, \gamma)$ is a quadratic Lie $\mathbb{K}$-superalgebra.*

### *6.2. The Alekseev–Meinrenken dynamical r-matrix*

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-superalgebra equipped with an invariant, even, bilinear form $\gamma : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{K}$.

To write our vCDYBE we need some preliminary notation. Recall (example 2.2) that $\gamma$ extends to a $S(\mathfrak{g})^*$-bilinear form on $\mathfrak{g}_x$. This extension, still denoted by $\gamma$, is also invariant.

Let $n \in \mathbb{N}$. With a differential form $\alpha \in A^n(\mathfrak{g}, \mathfrak{g})$ we associate $\hat{\alpha} \in A^{n+1}(\mathfrak{g})$ such that

$$\hat{\alpha}(\pi a_1, \ldots, \pi a_{n+1}) = \sum_{i=1}^{n+1} (-1)^{p(\pi a_i)(p(\pi a_1) + \cdots + p(\pi a_{i-1})) + p(\pi a_i) p(\alpha)} \gamma\left(a_1, \alpha(\pi a_1, \cdots, \widehat{\pi a_i}, \cdots, \pi a_{n+1})\right)$$

for any $a_1, \ldots, a_{n+1} \in \mathfrak{g}$. The degree of $\hat{\alpha}$ is $p(\hat{\alpha}) = p(\alpha) + 1$ modulo 2.

Let $\alpha \in A^1(\mathfrak{g}, \mathfrak{g})_1$ and $S \in A^3(\mathfrak{g})_1$, then $\mathrm{d}\alpha \in A^2(\mathfrak{g}, \mathfrak{g})_0$ and $[\alpha, \alpha] \in A^2(\mathfrak{g}, \mathfrak{g})_0$. It makes sense to consider the equation

$$\widehat{\mathrm{d}\alpha} - \tfrac{1}{2}\widehat{[\alpha, \alpha]} = S.$$

**Remark 6.1.** *If $S = 0$ this equation is analogous to the Maurer–Cartan equation in the version (15).*

Let $f(t) \in \mathbb{K}_0[[t]]$. We recall that $f(\operatorname{ad} x)(a) \equiv f^a$, $a \in \mathfrak{g}$, has been defined in (11): it is an element of the Lie $S(\mathfrak{g})^*$-superalgebra $\mathfrak{g}_x = \operatorname{Hom}(S(\mathfrak{g}), \mathfrak{g})$.

Let $\epsilon \in \mathbb{K}_0$. We want to solve the equation with $\alpha = \alpha_f \in A^1(\mathfrak{g}, \mathfrak{g})_1$ and $S = 4\epsilon T$ defined by

$$i(a)(\alpha_f) = (-1)^{p(a)} 2 f^a \qquad \text{for} \quad a, b, c, \in \mathfrak{g}$$
$$T(\pi a, \pi b, \pi c) = -(-1)^{p(b)} \gamma([a, b], c).$$

We have the odd differential form $T \in A^3(\mathfrak{g})_1$ because $\gamma$ is even, invariant, bilinear.

When $f(t) = -f(-t)$, the equation

$$\widehat{\mathrm{d}\alpha_f} - \tfrac{1}{2}\widehat{[\alpha_f, \alpha_f]} = 4\epsilon T \tag{18}$$

is our vCDYBE.

We want to write it in a different way. We start by considering the ring of formal power series $\mathbb{K}_0[[t, u, v]]$. For a monomial $t^i u^j v^k$ and $a, b, c \in \mathfrak{g}$, we introduce the object

$$(t^i u^j v^k : \gamma)(a, b, c) := \gamma\big((\operatorname{ad} x)^i(a), [(\operatorname{ad} x)^j(b), (\operatorname{ad} x)^k(c)]\big) \in S(\mathfrak{g})^*.$$

By linearity, we extend this definition to any $\phi \in \mathbb{K}_0[[t, u, v]]$ to obtain an element $(\phi(t, u, v) : \gamma)(a, b, c) \in S(\mathfrak{g})^*$. It is easy to see that if $\phi$ is symmetric in $t, u, v$ then $(\phi(t, u, v) : \gamma)$ is an antisymmetric, even, 3-linear form on $\mathfrak{g}$ with values in $S(\mathfrak{g})^*$. In particular, by lemma 2.1 it gives an odd differential 3-form $(\phi(t, u, v) : \gamma)_x \in A^3(\mathfrak{g})_1$ such that

$$(\phi(t, u, v) : \gamma)_x(\pi a, \pi b, \pi c) = -(-1)^{p(b)}(\phi(t, u, v) : \gamma)(a, b, c) \qquad \text{for} \quad a, b, c \in \mathfrak{g}.$$

**Remark 6.2.** $T(\pi a, \pi b, \pi c) = (1 : \gamma)_x(\pi a, \pi b, \pi c)$.

**Lemma 6.2.** *Let* $\Delta f(t, u) = 2\big(\frac{f(t+u)-f(u)}{t} + \frac{f(t+u)-f(t)}{u}\big)$. *We have*

$$\big(\widehat{\mathrm{d}\alpha_f} - \tfrac{1}{2}\widehat{[\alpha_f, \alpha_f]}\big)$$
$$= (\Delta f(u, v) + \Delta f(t, v) + \Delta f(t, u) - 4(f(u)f(v) + f(t)f(v) + f(t)f(u)) : \gamma)_x.$$

**Proof.** Let $a, b, c \in \mathfrak{g}$. By definition we have

$$\begin{aligned}(\widehat{\mathrm{d}\alpha_f})(\pi a, \pi b, \pi c) &= \gamma\big(a, \mathrm{d}\alpha_f(\pi b, \pi c)\big) + (-1)^{p(\pi b)p(\pi a)}\gamma\big(b, \mathrm{d}\alpha_f(\pi a, \pi c)\big)\\ &\quad + (-1)^{p(\pi c)p(\pi a+\pi b)}\gamma(c, \mathrm{d}\alpha_f(\pi a, \pi b)).\end{aligned}$$

As $\mathrm{d}\alpha_f(\pi b, \pi c) = i(b)i(c)(\mathrm{d}\alpha_f)$, lemma 4.1 gives

$$\begin{aligned}\gamma(a, \mathrm{d}\alpha_f(\pi b, \pi c)) &= (-1)^{p(\pi b)}\gamma(a, (\Delta f(t, u) : [b, c])_x)\\ &\equiv (-1)^{p(\pi b)}(\Delta f(u, v) : \gamma)(a, b, c).\end{aligned}$$

We get that

$$\begin{aligned}(\widehat{\mathrm{d}\alpha_f})(\pi a, \pi b, \pi c) &= (-1)^{p(\pi b)}(\Delta f(u, v) : \gamma)(a, b, c) + (-1)^{p(\pi b)p(\pi a)+p(\pi a)}(\Delta f(u, v) : \gamma)(b, a, c)\\ &\quad + (-1)^{p(\pi c)p(\pi a+\pi b)+p(\pi a)}(\Delta f(u, v) : \gamma)(c, a, b)\\ &= -(-1)^{p(b)}(\Delta f(u, v) + \Delta f(t, v) : \gamma)(a, b, c)\\ &\quad + (-1)^{p(\pi a+\pi b)p(\pi c)+p(\pi a)+p(c)p(a)+p(c)p(b)}(\Delta f(t, u) : \gamma)(a, b, c)\\ &= (-1)^{p(\pi b)}(\Delta f(u, v) + \Delta f(t, v) + \Delta f(t, u) : \gamma)(a, b, c)\\ &= (\Delta f(u, v) + \Delta f(t, v) + \Delta f(t, u) : \gamma)_x(\pi a, \pi b, \pi c).\end{aligned}$$

In the last equality we use that $\Delta f(u, v) + \Delta f(t, v) + \Delta f(t, u)$ is symmetric in $t, u, v$.

In an analogue way we compute $\widehat{[\alpha_f, \alpha_f]}(\pi a, \pi b, \pi c)$: it is sufficient to replace $\mathrm{d}\alpha_f$ by $[\alpha_f, \alpha_f]$, and $\Delta f(t, u)$ by $4f(t)f(u)$. □

In particular, this lemma and remark 6.2 give that vCDYBE (equation (18)) is the equation

$$(\Delta f(u,v)+\Delta f(t,v)+\Delta f(t,u)-4(f(u)f(v)+f(t)f(v)+f(t)f(u))-4\epsilon : \gamma)_x = 0.$$

**Lemma 6.3.** *If $\phi(t,u,v)=0$ modulo $t+u+v$ then $(\phi(t,u,v):\gamma)=0$.*

**Proof.** Let $\varphi \in \mathbb{K}_0[[t,u,v]]$ be a monomial. The invariance of $\gamma$ and Jacobi's identity (9) give immediately $(\varphi(t,u,v)(t+u+v):\gamma)=0$. □

**Theorem 6.1.** *The series $f \in \mathbb{K}_0[[t]]$ verifies equation (18) for all Lie $\mathbb{K}$-superalgebra equipped with an even, invariant, bilinear form, if*

$$\begin{aligned}\frac{f(t+u)-f(u)}{t}&+\frac{f(u+t)-f(t)}{u}+\frac{f(v+t)-f(t)}{v}+\frac{f(t+v)-f(v)}{t}\\ &+\frac{f(u+v)-f(v)}{u}+\frac{f(v+u)-f(u)}{v}\\ &=2\big(f(t)f(u)+f(u)f(v)+f(v)f(t)\big)+2\epsilon \qquad \text{modulo} \quad t+u+v \end{aligned} \tag{19}$$

*in $\mathbb{K}_0[[t,u,v]]$.*

Recall that $f(t)=-f(-t)$, setting $v=-t-u$ we get

$$\begin{aligned}\frac{f(t+u)-f(u)}{t}&+\frac{f(u+t)-f(t)}{u}+\frac{f(u)+f(t)}{u+t}\\ &=f(t)f(u)-f(u)f(t+u)-f(u+t)f(t)+\epsilon.\end{aligned} \tag{20}$$

**Remark 6.3.** As $f(t)=-f(-t)$, this equation comes also from

$$\begin{aligned}\frac{f(t+u)-f(u)}{t}&+\frac{f(t+u)-f(t)}{u}+\frac{f(v+t)-f(t)}{v}\\ &=f(t)f(u)+f(t)f(v)+f(u)f(v)+\epsilon \qquad \text{modulo} \quad t+u+v.\end{aligned}$$

In particular, as $f(t)=-f(-t)$, vCDYBE can be written also as the equation

$$\begin{aligned}&\left(\frac{f(t+u)-f(u)}{t}+\frac{f(t+u)-f(t)}{u}+\frac{f(v+t)-f(t)}{v}:\gamma\right)_x\\ &\qquad =(f(t)f(v)+f(u)f(v)+f(t)f(u)+\epsilon:\gamma)_x.\end{aligned} \tag{21}$$

**Lemma 6.4.** *Let $\mathbb{K}\supseteq\mathbb{Q}$. Equation (19) has only one solution $f\in\mathbb{K}_0[[t]]$ such that $f(t)=-f(-t)$. When $\epsilon=\frac{1}{4}$ this solution is $f(t)=-\frac{1}{t}+\frac{1}{2}\coth\left(\frac{t}{2}\right)\in\mathbb{Q}[[t]]$.*

**Proof.** We suppose that $f(0)=0$ and we evaluate equation (20) in $u=0$. We get $f'(t)=-2\frac{f(t)}{t}-f(t)^2+\epsilon$. As $\mathbb{K}\supseteq\mathbb{Q}$, this equation has only one odd formal power series in $\mathbb{K}_0[[t]]$ as solution. □

**Theorem 6.2.** *Let $\mathbb{K}\supseteq\mathbb{Q}$. For any Lie $\mathbb{K}$-superalgebra equipped with an even, invariant bilinear form, the series $f(t)=-\frac{1}{t}+\frac{1}{2}\coth\left(\frac{t}{2}\right)\in\mathbb{Q}[[t]]$ is a solution of equation (18) with $\epsilon=\frac{1}{4}$.*

**Remark 6.4** (The classical dynamical Yang–Baxter equation)**.** Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, $\mathfrak{g}$ be a finite-dimensional Lie algebra, $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$.

We consider a map $r : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ such that it is analytic in an open set containing zero. For any $1 \leqslant i, j \leqslant 3$ with $i \neq j$, we use the standard notation $r_{ij} : \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ (for example, $r_{12} = r \otimes 1$ and $r_{23} = 1 \otimes r$). Let

$$CYBE(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}.$$

For any $a \in \mathfrak{g}$ we use the standard notation $a_1 = a \otimes 1 \otimes 1$, $a_2 = 1 \otimes a \otimes 1$, $a_3 = 1 \otimes 1 \otimes a$. Let $\{e_j\}_j$ be a basis of $\mathfrak{h}$, we introduce (see [6])

$$CDYBE(r) := CYBE(r) + \sum_j (e_j)_1 \frac{\partial r_{23}}{\partial e_j} - (e_j)_2 \frac{\partial r_{13}}{\partial e_j} + (e_j)_3 \frac{\partial r_{12}}{\partial e_j} \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}.$$

We say that $r$ verifies the classical dynamical Yang–Baxter equation if $CDYBE\,(r) = 0$.

Let us suppose that $\mathfrak{g}$ is equipped with a $\mathbb{K}$-valued bilinear form $\gamma$ such that $(\mathfrak{g}, \gamma)$ is a quadratic Lie algebra. We identify $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. The bilinear form $\gamma$ defines an element $c \in \mathfrak{g} \otimes \mathfrak{g}$. By looking for solutions for which $r - r_{21}$ is a constant multiple of $c$, one obtains a modified version of the classical dynamical Yang–Baxter equation (vCDYBE) for the antisymmetric part of $r$. We denote by $\{e^i\}_i$ the basis for $\mathfrak{g}$ such that $\gamma(e^i, e_j) = \delta_{i,j}$ for any $i, j$. Let $\varepsilon \in \mathbb{K}$ and $\varphi := e^j \otimes e^k \otimes [e_j, e_k] \in \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$. We consider $\mathfrak{h} = \mathfrak{g}$. The modified equation vCDYBE with coupling constant $\varepsilon$ is the equation $CDYBE\,(\tilde{r}) = \varepsilon\varphi$, where $\tilde{r} : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, is a function with values in the antisymmetric part of $\mathfrak{g} \otimes \mathfrak{g}$.

Let $f(t) \in \mathbb{K}[[t]]$ verify $f(t) = -f(-t)$, we consider

$$r_\gamma = \sum_{j,k} e^j \otimes e^k \gamma(e_j, f^{e_k}).$$

Using the properties of $\gamma$, we get that the modified equation vCDYBE for $r_\gamma$ is equivalent to

$$\begin{aligned} &\gamma([f^c, b], f^a) + L(c)(\gamma(f^a, b)) + \gamma([f^b, a], f^c) - L(b)(\gamma(f^a, c)) \\ &\qquad + \gamma([f^a, c], f^b) + L(a)(\gamma(f^b, c)) = \epsilon\gamma([a, b], c) \qquad \forall a, b, c \in \mathfrak{g}. \end{aligned}$$

This cyclotomic equation is equation (21) for $\mathfrak{g} = \mathfrak{g}_0$.

**Remark 6.5.** Note that this cyclotomic equation makes sense because $f(t) = -f(-t)$ and because $\gamma$ is an invariant, symmetric, even, bilinear form. In fact, this gives

$$\begin{aligned} &\gamma(f(\operatorname{ad} x)(a), b) = \gamma(a, f(-\operatorname{ad} x)(b)) = -\gamma(a, f(\operatorname{ad} x)(b)) \\ &\gamma(f^a, b) = \gamma(b, f^a) \end{aligned}$$

in $S(\mathfrak{g})^*$, so the formula $\gamma(f^a, b)$ defines an antisymmetric bilinear form on $\mathfrak{g}$ with values in $S(\mathfrak{g})^*$. Using lemma 2.1, there exists a unique even differential 2-form $\omega_f \in A^2(\mathfrak{g})_0 = \operatorname{Hom}(S^2(\Pi\mathfrak{g}) \otimes S(\mathfrak{g}), \mathbb{K})_0 = \operatorname{Hom}(S^2(\Pi\mathfrak{g}), S(\mathfrak{g})^*)_0$ such that

$$i(a)i(b)(\omega_f) = \omega_f(\pi a, \pi b) = \gamma(f^a, b) \qquad \forall a, b \in \mathfrak{g}.$$

**Definition 6.2.** *A map $r$ solution of vCDYBE is called a dynamical r-matrix if $r^{21} + r$ is symmetric and $\mathfrak{g}$-invariant, $r : \mathfrak{g}^* \equiv \mathfrak{h}^* \to \mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{h}$ invariant.*

**Remark 6.6.** Etingof and Varchenko in [6] classify the solutions of the CDYBE when $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra.

Alekseev and Meinrenken in [1] consider the modified equation vCDYBE with $\epsilon = \frac{1}{4}$ and $\mathfrak{g}$ a compact Lie algebra. They show that $f(t) = -\frac{1}{t} + \frac{1}{2}\coth\left(\frac{t}{2}\right)$ gives a dynamical $r$-matrix (they deduce it using [6]).

The functional equation (20) was found independently by Fehér and Pusztai in [7]. They give another direct proof of the fact that $r_\gamma$ is a solution of vCDYBE when

$f(t) = -\frac{1}{t} + \frac{1}{2}\coth\left(\frac{t}{2}\right)$. Their proof uses the theory of holomorphic functional calculus of linear operators.

Different proofs, not involving the functional equation (20), are given in the appendix of [5] and in [4].

**Remark 6.7.** Let $\mathbb{K} = \mathbb{K}_0$ be a field of characteristic zero. As theorem 6.2 is verified by any Lie $\mathbb{K}$-superalgebra, some results of [1] are valid also for a Lie $\mathbb{K}$-superalgebra. In particular, the existence of the quantization map introduced in [1] and its intertwining properties hold for a Lie $\mathbb{K}$-superalgebra. See the PhD thesis [9] for more details.

## Acknowledgments

I am grateful to my advisor Michel Duflo for helpful discussions. In particular, he had the idea of writing the modified CDYBE as equations (18) and (21). I also thank the 'Section de Mathématiques' of Geneva for their hospitality, and the Department of Cergy-Pontoise for having organized my timetable in order to make possible my visit to Geneva.

## References

[1] Alekseev A and Meinrenken E 2000 The non-commutative Weil algebra *Invent. Math.* **139** 135–72

[2] Alekseev A and Meinrenken E 2003 Clifford algebras and the classical dynamical Yang–Baxter equation *Math. Res. Lett.* **10** 253–68 (*Preprint* math. RT/0209347)

[3] Bahturin Y A, Mikhalev A A, Petrogradsky V M and Zaicev M V 1992 *Infinite-Dimensional Lie Superalgebras* (Berlin: de Gruyter)

[4] Enriquez B and Etingof P 2003 Quantization of Alekseev–Meinrenken dynamical *r*-matrices *Preprint* math. QA/0302067

[5] Etingof P and Schiffmann O 2001 On the moduli space of classical dynamical *r*-matrices *Math. Res. Lett.* **8** 157–70

[6] Etingof P and Varchenko A 1998 Geometry and classification of solutions of the classical dynamical Yang–Baxter equation *Commun. Math. Phys.* **192** 77–120

[7] Fehér L and Pusztai B G 2001 A note on a canonical dynamical *r*-matrix *J. Phys. A: Math. Gen.* **34** 7335–48

[8] Leites D A 1980 Introduction to the theory of supermanifold *Russ. Math. Surv.* **35** 1–64

[9] Petracci E 2003 Functional equations and Lie algebras *Tesi di Dottorato* Università di Roma 'La Sapienza'

[10] Petracci E 2003 Universal representations of Lie algebras by coderivations *Bull. Sci. Math.* **127** 439–65

[11] Sharpe R W 1997 Differential Geometry, Cartan's Generalization of Klein's Erlangen Program *(Graduate Texts in Mathematics* vol 166*)* (Berlin: Springer)